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A Composite Mixed Finite Elements for General Hexahedral Grids*

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Abstract: A new mixed finite element method on three-dimensional general hexahedral meshes for second order elliptic problems is proposed. This finite element is composite and is shown to have optimal convergence properties.

Key-words: mixed finite element, hexahedral grids

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Un élément fini mixte composite pour des maillages hexaèdriques généraux

Résumé : Un nouvel élément fini mixte pour des maillages tridimensionnels hexaèdriques généraux est proposé. C'est un élément fini composite pour lequel les propriétés de convergence optimale sont démontrées.

Mots-clés : élément fini mixte, maillages héxaèdriques

1 Introduction

Single phase incompressible flow in a porous medium is governed by the Darcy flow equation, an elliptic equation coupling a conservation equation with Darcy's law. If gravity is neglected the mixed form of this equation becomes the system

$$\begin{aligned} \mathbf{u} &= -K\,\mathbf{grad}\ p \quad \text{in} \quad \Omega \\ \operatorname{div} \mathbf{u} &= f \qquad\qquad \text{in} \quad \Omega, \end{aligned}$$

where the primary unknown p is the fluid pressure, the secondary unknown $\mathbf{u}$ is the Darcy flow velocity, the coefficient K is a piecewise constant symmetric tensor, and f is a source term. It has been known since the early 80's that mixed finite element methods are particularly useful for the numerical simulation of this system of equations. There are several reasons why this is so. First of all, with a mixed method the Darcy velocity $\mathbf{u}$ is calculated simultaneously with the pressure p and to the same order of accuracy. Also in many applications the permeability K is discontinuous and can vary over several orders of magnitude from one geological region to another, and mixed methods are particularly suited to handling this difficulty. Another advantage is that mixed methods can handle easily nonrectangular meshes and nondiagonal tensors K. Finally, equally important is the fact that mixed methods are conservative and even locally conservative and for most geophysical applications this is an essential feature.

Many mixed methods for second order elliptic problems were introduced. Among the most well known of these are [16], [14], [15], [5], [4], and [3]. These elements are all based on triangular or rectangular elements in 2 dimensions or on tetrahedral, parallelepiped, or prismatic elements in 3 dimensions. For large calculations, regular meshes of rectangular or parallelepipedic elements are particularly efficient. However, for geophysical applications, the porous medium is a geological structure and is not always well suited to a regular mesh of rectangular elements. A natural idea is to deform a regular rectangular mesh so that the elements are convex quadrilaterals or hexahedra and to construct finite elements on the mesh by using multilinear mappings to a reference rectangle or rectangular solid. The approximation space on the deformed element is the image under the Piola transformation (see [12]) of the approximation space on the reference element. However, unless the new elements are parallelograms or parallelepipeds, so that the multilinear maps are actually affine, the classical scaling arguments break down and interpolation accuracy is lost. Though there remain problems for two dimensional elements, see [7], the problem is particularly evident for three dimensional elements. We describe an example due to Tom Russell given in [13] to show that if the approximation space on the reference element is $\mathbf{RTN}_0$, the lowest order Raviart Thomas Nédélec space, cf. [14, 17], the resulting approximation space does not even contain the constant functions. Consider the truncated pyramid E of unit height and with square horizontal bases of extents $s_0 \times s_0$ and $s_1 \times s_1$, shown below in Figure 1.1, and suppose that the constant vector field $\mathbf{u}(x) = (0,0,1)^t, \forall x \in E$, does belong to the approximation space. The exact flux through a horizontal section B_z, for $0 \le z \le 1$ is equal to the area of this section

$$\int_{B_z} \mathbf{u} \cdot \mathbf{n}_z = ((1-z)s_0 + zs_1)^2.$$

However the flux of any $\mathbf{v} \in \mathbf{RTN}_0$ of the unit reference cube through a horizontal section $\hat{B}_z$ varies linearly with z

$$\int_{\hat{B}_z} \mathbf{v} \cdot \mathbf{n}_z = (1-z)s_0^2 + zs_1^2.$$

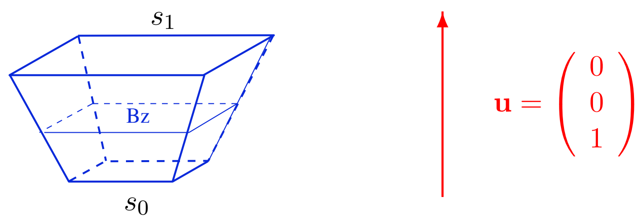

Figure 1.1: A hexahedron in the form of a truncated pyramid and a constant flow field

Since the Piola transformation preserves flux through any section, the constant field $\mathbf{u}$ can not be the image of any vector field in $\mathbf{RTN}_0$.

To show the effect in an actual computation we show a calculation carried out by Martin Prosi [1]. The calculation domain is a right circular cylinder. The permeability is constant and the source term is null. The boundary conditions imposed on the sides of the cylinder are so called no flow conditions, i.e. homogeneous Neumann conditions, and a constant pressure is given on each of the two bases of the cylinder with a pressure drop from one end to the other. Thus for the analytic solution the pressure is constant on each cross section parallel to the axis and it varies linearly from one end to the other while the flow field is constant and is parallel to the axis of the cylinder. The results shown in Figure 1.2 were obtained with deformed $\mathbf{RTN}_0$ elements. The pressure result is correct, but the flow field is not constant.

Several articles have addressed the problem of defining a mixed finite element on a distorted, nonparallelogram rectangle or on a distorted, nonparallelepiped rectangular solid at least for lowest order elements. In [19] and [1, 7] elements are introduced for convex quadrilateral elements, i.e. two dimensional elements but neither of these has a satisfactory extension to convex hexahedral elements, or three dimensional elements. In [10] a composite element was introduced for convex quadrilaterals in which the quadrilateral was subdivided into two triangles. The space of functions corresponding to the subdivided quadrilateral was the space of $\mathbf{H}(\text{div})$-functions on the quadrilateral such that the restriction to each of the two triangles was in $\mathbf{RTN}_0$ of the triangle and such that the divergence of the function was constant over the entire quadrilateral. In a paper of Kuznetsov and Repin [11] this idea was extended in a general way to elements that are three dimensional polygons.

In this article, following the ideas of Kuznetsov and Repin, we develop a composite element specifically for a convex hexahedron. This element is obtained by dividing the hexahedron into five tetrahedra and it is shown to have optimal convergence properties. In particular, unlike in [11], no extra regularity on the solution is needed. Also, unlike in [11], the analysis given here does not require that the set of all tetrahedra obtained from dividing the hexahedra form a mesh. This is important because it is not always possible to obtain a tetrahedral mesh from a general hexahedral mesh by subdividing the hexahedrons into five tetrahedra. In certain cases to obtain a tetrahedral mesh some of the hexahedra must be divided into six tetrahedra.

We will assume throughout this article that the boundary $\partial\Omega$ of Ω is made up of a nonempty part Γ_D on which a Dirichlet boundary condition is imposed and that on the

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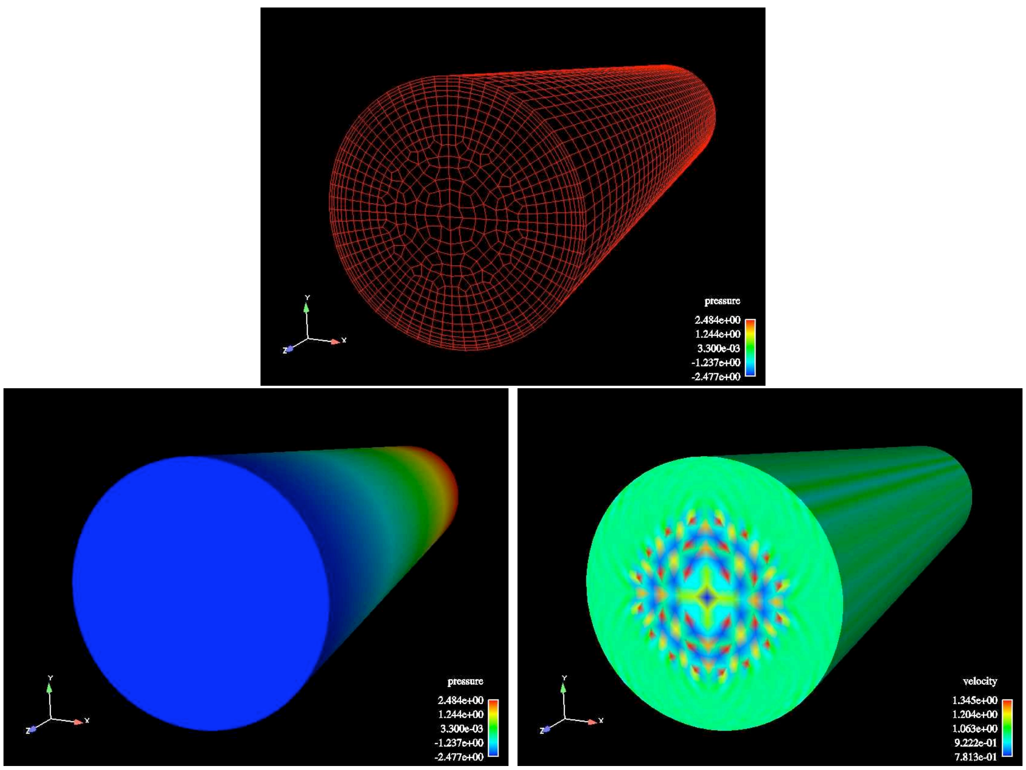


Figure 1.2: Difficulty for a mesh of nonparallelepiped hexahedra

remainder or the boundary Γ_N, a homogeneous Neumann condition has been imposed.

$$(\mathcal{P}) \qquad \begin{array}{lcllc} \mathbf{u} & = & -K\,\mathbf{grad}\; p & \text{in} & \Omega, \\ \operatorname{div}\mathbf{u} & = & f & \text{in} & \Omega, \\ p & = & p_d & \text{on} & \Gamma_D, \\ \mathbf{u}\cdot\mathbf{n}_\Omega & = & 0 & \text{on} & \Gamma_N. \end{array} \tag{1.1}$$

Where $\mathbf{n}_\Omega$ is the unit outward normal vector on the boundary of Ω. Throughout we will use the notation, if X is a domain in $\mathbb{R}^3$, then $\mathbf{n}_X$ is a unit, outward pointing normal vector field on the boundary of X. If Y is part of the boundary of a domain in $\mathbb{R}^3$, then $\mathbf{n}_Y$ denotes a unit normal vector field on Y.

In Section 2 we recall some of the theory for mixed finite elements methods. The new mixed finite element approximation is developed in Section 3. Section 4 is devoted to the interpolation error. In Section 5 we present some numerical results.

2 Numerical analysis for mixed methods

In this section we recall some well known results for mixed finite element methods.
In [2] it is shown that if $\mathcal{W}$ and $\mathcal{M}$ are Hilbert spaces and if $a : \mathcal{W}\times\mathcal{W} \longrightarrow \mathbb{R}$ and $b : \mathcal{W}\times\mathcal{M} \longrightarrow \mathbb{R}$, are continuous bilinear forms satisfying the following two conditions:

$$(i) \quad a \text{ is } \mathcal{V}\text{-elliptic, where } \mathcal{V} = \{\mathbf{v}\in\mathcal{W} \,:\, b(\mathbf{v},q) = 0, \quad \forall q\in\mathcal{M}\},\ \text{i. e.} \tag{2.1}$$

$$\exists \alpha > 0 \text{ such that } \forall \mathbf{v}\in\mathcal{V}, \quad a(\mathbf{v},\mathbf{v}) \geq \alpha\|\mathbf{v}\|^2_{H(\mathrm{div};\Omega)}$$

$$(ii) \quad b \text{ satisfies the inf-sup condition on } \mathcal{W}\times\mathcal{M},\ \text{i. e.} \tag{2.2}$$

$$\exists \beta > 0 \text{ such that } \inf_{q\in\mathcal{M}}\sup_{\mathbf{v}\in\mathcal{W}} b(\mathbf{v},q) \geq \beta\|\mathbf{v}\|_{H(\mathrm{div};\Omega)}\|q\|_{L^2(\Omega)},$$

then if $\mathcal{L}_\mathcal{W} : \mathcal{W} \longrightarrow \mathbb{R}$ and $\mathcal{L}_\mathcal{M} : \mathcal{M} \longrightarrow \mathbb{R}$ are continuous linear forms there exists a unique solution ($\mathbf{u}$, p) to the problem

$$(\mathcal{P}_w) \qquad \begin{array}{l} \text{Find } \mathbf{u}\in\mathcal{W} \text{ and } p\in\mathcal{M} \text{ such that} \\ a(\mathbf{u},\mathbf{v}) - b(\mathbf{v},p) = \mathcal{L}_\mathcal{W}(\mathbf{v}), \quad \forall \mathbf{v}\in\mathcal{W} \\ b(\mathbf{u},q) = \mathcal{L}_\mathcal{M}(q), \quad \forall q\in\mathcal{M}. \end{array} \tag{2.3}$$

If also $\mathcal{W}_h$ and $\mathcal{M}_h$ are finite element subspaces of $\mathcal{W}$ and $\mathcal{M}$ respectively, and the bilinear forms a and b are such that

$$(i) \quad a \text{ is } \mathcal{V}_h\text{-elliptic, where } \mathcal{V}_h = \{\mathbf{v}\in\mathcal{W}_h \,:\, b(\mathbf{v},q) = 0, \quad \forall q\in\mathcal{M}_h\},\ \text{i. e.} \tag{2.4}$$

$$\exists \alpha_h > 0 \text{ such that } \forall \mathbf{v}\in\mathcal{V}_h, \quad a(\mathbf{v},\mathbf{v}) \geq \alpha_h\|\mathbf{v}\|^2_{H(\mathrm{div};\Omega)}$$

$$(ii) \quad b \text{ satisfies the inf-sup condition on } \mathcal{W}_h\times\mathcal{M}_h,\ \text{i. e.} \tag{2.5}$$

$$\exists \beta_h > 0 \text{ such that } \inf_{q\in\mathcal{M}_h}\sup_{\mathbf{v}\in\mathcal{W}_h} b(\mathbf{v},q) \geq \beta_h\|\mathbf{v}\|_{H(\mathrm{div};\Omega)}\|q\|_{L^2(\Omega)},$$

then the discretized problem

$$(\mathcal{P}_h) \quad \begin{array}{l} \text{Find } \mathbf{u} \in \mathcal{W}_h \text{ and } p \in \mathcal{M}_h \text{ such that} \\ a(\mathbf{u},\mathbf{v}) - b(p,\mathbf{v}) = \mathcal{L}_{\mathcal{W}}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{W}_h \\ b(q,\mathbf{u}) = \mathcal{L}_{\mathcal{M}}(q) \quad \forall q \in \mathcal{M}_h, \end{array} \tag{2.6}$$

admits a unique solution $(\mathbf{u}_h,p_h)$, and

$$\|p-p_h\|^2_{L^2(\Omega)} + \|\mathbf{u}-\mathbf{u}_h\|^2_{H(\mathrm{div};\Omega)} \le C \left\{ \inf_{q_h \in \mathcal{M}_h} \|p-q_h\|^2_{L^2(\Omega)} + \inf_{\mathbf{v}_h \in \mathcal{W}_h} \|\mathbf{u}-\mathbf{v}_h\|^2_{H(\mathrm{div};\Omega)} \right\}$$

with a constant C which depends only on the constants of continuity of the bilinear forms a and b and the constants α_h and β_h.

With $\mathcal{W}$=$\mathbf{H}(\mathrm{div};\Omega)$ and $\mathcal{M}$=$L^2(\Omega)$ and

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{v}\cdot\mathbf{u} \qquad \text{and} \qquad b(\mathbf{u},q) = \int_\Omega q \,\mathrm{div}\,\mathbf{v},$$

problem $(\mathcal{P}_w)$ is the weak form of $(\mathcal{P})$ where the forms $\mathcal{L}_{\mathcal{W}} : \mathcal{W} \longrightarrow \mathbb{R}$ and $\mathcal{L}_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathbb{R}$ are determined by the source term and the Dirichlet boundary data respectively. Further a and b satisfy the conditions (2.1) and(2.2) so there is a unique solution $(\mathbf{u},p)$ to problem $(\mathcal{P})$.

Thus to see that a pair of spaces $(\mathcal{W}_h,\mathcal{M}_h)$ is suitable for a mixed method it suiffices to show that the two conditions (2.4) and (2.5) are satisfied with contants α_h and β_h independent of h and to estimate interpolation errors.

For the spaces we shall consider, $(\mathcal{W}_h,\mathcal{M}_h)$ it is easy to see that the condition (2.4) is satisfied because $\mathcal{V}_h \subset \mathcal{V}$ so that α_h may be taken to be α. For the condition (2.5) we know that $\mathcal{M}_h \subset \mathcal{M}$ and b satisfies (2.2) for the spaces $(\mathcal{W},\mathcal{M})$. Thus given $q_h \in \mathcal{M}_h$ there is an element $\mathbf{v} \in \mathcal{W}$ such that $b(\mathbf{v},q_h) \ge \beta\|\mathbf{v}\|_{H(\mathrm{div};\Omega)}\|q_h\|_{L^2(\Omega)}$. Thus following standard procedure we define a projection operator Π_h onto $\mathcal{W}_h$ such that $b(\Pi_h\mathbf{v},q_h) = b(\mathbf{v},q_h)$, $\forall q_h \in \mathcal{M}_h$ and such that $\|\Pi_h\mathbf{v}\|_{H(\mathrm{div};\Omega)} \le \gamma_h\|\mathbf{v}\|_{H(\mathrm{div};\Omega)}$ and show that γ_h is in fact independent of h when h is sufficiently small. We will also give interpolation estimates.

3 The approximation spaces

Suppose that Ω is a convex polyhedral domain in $\mathbb{R}^3$ and let $\mathcal{M} = L^2(\Omega)$ and $\mathcal{W} = \{\mathbf{u} \in \mathbf{H}(\mathrm{div};\Omega) \ : \ \mathbf{u}\cdot\mathbf{n}_\Omega = 0 \text{ on } \Gamma_N\}$. Let $\mathcal{T}_h$ be a mesh made up of general convex hexahedral cells E with planar faces such that the permeability K is constant on each cell E. Let $\mathcal{F}_h$ be the set of faces F (quadralateral) of elements of $\mathcal{T}_h$. Let $\mathcal{M}_h \subset \mathcal{M}$ be the space of L^2 functions on Ω that are constant on each cell $E \in \mathcal{T}_h$. The object of this section is to define an approximation space $\mathcal{W}_h \subset \mathcal{W}$ such that the pair $(\mathcal{W}_h,\mathcal{M}_h)$ yields a pair of approximation spaces appropriate for mixed finite element approximation in the sense that both Brezzi conditions are satisfied and that the spaces permit order h approximation of sufficiently regular functions.

The approximation space $\mathcal{W}_h \subset \mathcal{W}$ will be defined such that an element $\mathbf{u}_h \in \mathcal{W}_h$

- has constant divergence in each hexahedron $E \in \mathcal{T}_h$,
- has constant normal component on each face $F \in \mathcal{F}_h$,
- is uniquely determined by its normal traces on the faces $F \in \mathcal{F}_h$.

as is the case for the Raviart-Thomas-Nédélec mixed finite element approximation space of lowest-order for a tetrahedral or a parallelepiped mesh. Toward this end, we construct a composite element by subdividing each hexahedron E into 5 tetrahedra $T_j^E, \quad j = 1, \cdots, 5$.

We define a space $\mathcal{W}_E$ for each E in $\mathcal{T}_h$ and we will define $\mathcal{W}_h$ the space of $\mathbf{H}(\mathrm{div};\Omega)$ - functions that are locally in $\mathcal{W}_E$. Thus the pair of approximation spaces is defined by

$$\begin{aligned} \mathcal{M}_h &= \{q \in \mathcal{M} \, : \, q_{|E} \text{ is constant } \forall E \in \mathcal{T}_h\}, \\ \mathcal{W}_h &= \{\mathbf{u} \in \mathcal{W} \, : \, \mathbf{u}_{|E} \in \mathcal{W}_E \; \forall E \in \mathcal{T}_h\}. \end{aligned}$$

3.1 A composite element

One begins by choosing either set of 4 of the 8 vertices for which no pair are joined by an edge of a face of E; i.e. one chooses any of the 8 verticies together with the three vertices which shave a face with the original vertex but are not joined to it by an edge. These four points determine a tetrahedron T_5^E whose faces are in the interior of E, and $E \setminus T_5^E$ is made up of four disjoint tetrahedra $T_j^E, \quad j = 1, \cdots, 4$

$$E \;=\; T_1^E \cup T_2^E \cup T_3^E \cup T_4^E \cup T_5^E,$$

$$T_1^E = ABDE, \quad T_2^E = FGBE, \quad T_3^E = CDBG, \quad T_4^E = HGED, \quad T_5^E = BDEG$$

as shown in Figure 3.1. Each of the first four tetrahedra, $T_j^E \quad j = 1, \cdots, 4$, has three

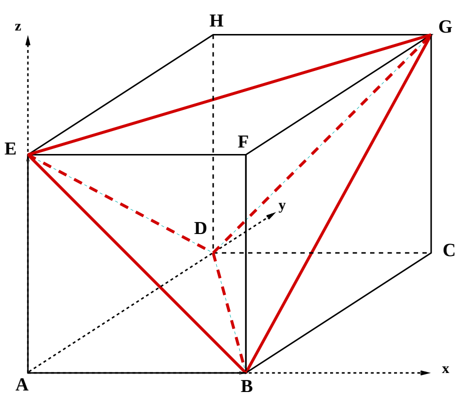


Figure 3.1: One partition of the reference hexaedron into 5 tetrahedra

triangular faces which are contained in the boundary of E and one internal face. (In fact if E is a cube these four tetrahedra are similar). The four internal faces of these four tetrahedra form the boundary of the fifth tetrahedron T_5^E. Note that for a given hexahedron there are exactly two ways to subdivide it in such a manner.

We denote by $\widetilde{\mathcal{T}}_E$ one of the triangulations of E made up of these 5 tetrahedra:

$$\widetilde{\mathcal{T}}_E = \{T_j^E : j = 1, \cdots, 5\}.$$

We will also use the notation $\mathcal{F}_E$ for the set of faces F of the hexahedron E and $\widetilde{\mathcal{F}}_E$ for the set of faces $\widetilde{F}$ of the tetrahedra $T \in \widetilde{\mathcal{T}}_E$:

$$\mathcal{F}_E = \{F \in \mathcal{F}_h \;:\; F \text{ is a face of } E\}, \qquad \widetilde{\mathcal{F}}_E = \{\widetilde{F} \;:\; \widetilde{F} \text{ is a face of } T_j^E \text{ for some } j, j = 1, \cdots 5\}.$$

Note that $\mathcal{F}_E$ has 6 elements and $\widetilde{\mathcal{F}}_E$ 16 elements, 12 of which are on the boundary of E and 4 of which lie in the interior of E.

3.2 Local approximation spaces

Here we define several local approximation spaces both for scalar functions and for vector valued functions. For a given hexahedron $E \in \mathcal{T}_h$ define the space $\mathcal{M}_E$ to be the space of scalar functions q that are constant on E and let $\widetilde{\mathcal{M}}_E$ denote the space of piecewise constant functions on E that are constant on each tetrahedra of $\widetilde{\mathcal{T}}_E$:

$$\begin{aligned} \mathcal{M}_E &= \{q \in L^2(E) \ : \ q_{|E} \text{ is constant } \}, \\ \widetilde{\mathcal{M}}_E &= \{q \in L^2(E) \ : \ q_{|T_j^E} \text{ is constant}, \ j = 1, \cdots, 5\}, \end{aligned}$$

and note that

$$\mathcal{M}_E \subset \widetilde{\mathcal{M}}_E.$$

An element of $\mathcal{M}_E$ is determined by its constant value on E and an element of $\widetilde{\mathcal{M}}_E$ is determined by its constant values on the 5 tetrahedra of $\widetilde{\mathcal{T}}_E$.

The space associated with the composite element is $\mathcal{W}_E$, the space of vector functions $\mathbf{v} \in \mathbf{H}(\text{div}; E)$ satisfying the following conditions:

$$\mathbf{v}_{|T_j^E} \in \mathbf{RTN}_0(T_j^E), \quad j = 1, \cdots, 5, \tag{3.1}$$

$$\text{div } \mathbf{v} \quad \text{is a constant over E}, \tag{3.2}$$

$$\mathbf{v} \cdot \mathbf{n}_E \quad \text{is constant on each face of E}. \tag{3.3}$$

We denote by $\widetilde{\mathcal{W}}_E$ the lowest order Raviart-Thomas-Nedelec space over E associated with the discretisation $\widetilde{\mathcal{T}}_E$ of E and define the intermediate space $\widehat{\mathcal{W}}_E$ to be the elements of $\widetilde{\mathcal{W}}_E$ having constant divergence so that

$$\begin{aligned} \widetilde{\mathcal{W}}_E &= \{\mathbf{v} \in \mathbf{H}(\text{div}; E) \ : \ \mathbf{v}_{|T_j^E} \in \mathbf{RTN}_0(T_j^E), \quad j = 1, \cdots, 5\}, \\ \widehat{\mathcal{W}}_E &= \{\tilde{\mathbf{v}} \in \widetilde{\mathcal{W}}_E \ : \ \text{div } \tilde{\mathbf{v}}|_E \text{ is constant}\}, \\ \mathcal{W}_E &= \{\hat{\mathbf{v}} \in \widehat{\mathcal{W}}_E \ : \ \hat{\mathbf{v}} \cdot \mathbf{n}_{E\,|F} \text{ is constant}, \forall F \in \mathcal{F}_E\} \end{aligned}$$

and

$$\mathcal{W}_E \subset \widehat{\mathcal{W}}_E \subset \widetilde{\mathcal{W}}_E.$$

It is well known that an element of $\widetilde{\mathcal{W}}_E$ is uniquely determined by the constant values of the normal fluxes through the faces $\widetilde{F} \in \widetilde{\mathcal{F}}_E$ and that a basis for $\widetilde{\mathcal{W}}_E$ consists of the set of functions $\tilde{\omega}_{\tilde{F}_i}$, $\widetilde{F}_i \in \widetilde{\mathcal{F}}_E$ having constant normal component on the face $\widetilde{F}_j$ equal to $\delta_{i,j}, j = 1, \cdots, 16$.

To check that a function $\mathbf{v}$ of $\mathcal{W}_E$ is uniquely defined by its normal traces through the 6 faces of E, first note that it is also in $\widetilde{\mathcal{W}}_E$ and so it is determined by the 16 degrees of freedom which are its normal traces on the faces in $\widetilde{\mathcal{F}}_E$. However since its normal traces on the faces in $\mathcal{F}_E$ are constant, the number of degrees of freedom is reduced to 10. (Each (quadralateral) face in $\mathcal{F}_E$ is made up of 2 (triangular) faces in $\widetilde{\mathcal{F}}_E$.) Since the divergence of the element is the same on each of the five tetrahedra in $\widetilde{\mathcal{T}}_E$, the number of degrees of freedom is reduced again by 4. Thus there remain six degrees of freedom to be determined. To check for unisolvence it suffices to note that with g_i denoting the function defined on the boundary of E by $g_{i\,|F_j} = \delta_{i,j}$ for each face $F_j, j = 1, \cdots, 6$, of E, ω_i is defined to be the component $\mathbf{u}$ of the solution $(\mathbf{u}, p)$

to the mixed finite element problem

$$\text{Find } (\mathbf{u}, p) \in (\widetilde{\mathcal{W}}_E^{g_i}, \widetilde{\mathcal{M}}_E) \text{ such that}$$

$$(\mathcal{P}_E) \qquad \begin{array}{rcll} a(\mathbf{u},\mathbf{v}) - b(\mathbf{v},p) & = & 0 & \forall \mathbf{v} \in \widetilde{\mathcal{W}}_E^0 \\ b(\mathbf{u},q) & = & \dfrac{|F_i|}{|E|} & \forall q \in \widetilde{\mathcal{M}}_E, \end{array}$$

where $\widetilde{\mathcal{W}}_E^{g_i}$, respectively $\widetilde{\mathcal{W}}_E^0$, is the subspace of $\widetilde{\mathcal{W}}_E$ consisting of those elements whose normal traces agree with g_i, respectively are equal to 0, on the boundary of E. Note that while this Neumann problem determines p only up to a constant, it determines $\mathbf{u}$ uniquely so that ω_i is well defined. To see that this is true suppose that ω_i' is a second such solution. Then the difference $\omega_i - \omega_i'$ is a solution to the problem $(\mathcal{P}_E)$ but with the boundary term g_i replaced by 0 and the source term $\dfrac{|F_i|}{|E|}$ replaced by 0. Thus the solution has zero divergence in each of the tetrahedra of $\mathcal{T}_E$ and has zero normal component on all of the external faces of $\widetilde{F}_E$. But each of the exterior tetrahedra $T_i^E, i = 1, \cdots, 4$ has three faces on the boundary of E. Since the flux through each of these faces is null and since the divergence on the tetrahedron is null, the flux through the fourth face, the interior face must also be null. Thus $\omega_i = \omega_i'$. The set of functions $\omega_i, i = 1, \cdots, 6$, thus defined forms a basis of $\mathcal{W}_E$.

In the same way one can see that an element of $\widehat{\mathcal{W}}_E$ is determined by the values of its normal components on the 12 faces in $\widetilde{\mathcal{F}}_E$ that lie on the boundary of E, and that a basis for $\widehat{\mathcal{W}}_E$ is made up of the 12 functions $\hat{\omega}_i$ defined in the obvious maner.

Remark 1 *It might seem more natural to use the canonical subdivision of the hexahedron into 6 tetrahedra, (all of which are identical when the hexahedron is a cube). If instead, the hexahedron were divided in this way into 6 tetrahedra, there would be 18 coefficients to determine corresponding to 18 tetrahedral faces. Conditions (3.2) and (3.3) impose 5 and 6 constraints respectively leaving 7 degrees of freedom to calculate. The macroelement would not be unisolvant. One could still solve a problem analagous to $(\mathcal{P}_E)$ but the element ω_i would no longer be uniquely determined as with this alternative decomposition each tetrahedron would have two external faces and two internal faces. This decomposition introduces an edge which does not lie on the boundary of E and around which a divergence free flow could turn. As was suggested to us by Todd Arbogast, one could instead work with the rotational free subspace to obtain the right dimension for the approximation space.*

Remark 2 *Note that while $\mathcal{W}_h$ and $\mathcal{M}_h$ are defined from the local spaces $\mathcal{W}_E$ and $\mathcal{M}_E$ global spaces $\widetilde{M}_h$, $\widetilde{W}_h$ and $\widehat{W}_h$ can not be defined since the decomposition of the elements E is not necessarily done in a way that makes the set of all T such that $T \in \mathcal{T}_E$ for some $E \in \mathcal{T}_h$ a triangulation of Ω.*

4 Interpolation error

As we saw in Section 2 it follows from Babuska-Brezzi theory that the errors committed in using the mixed finite element method with approximation spaces satisfying the 2 conditions (2.1), (2.2) is of the same order as the error of interpolation:

$$\|p - p_h\|_{L^2(\Omega)} + \|\mathbf{u} - \mathbf{u}_h\|_{H(\mathrm{div};\Omega)} \le C \left(\inf_{q_h \in L^2(\Omega)} \|p - q_h\|_{L^2(\Omega)} + \inf_{\mathbf{v}_h \in H(\mathrm{div},\Omega)} \|\mathbf{u} - \mathbf{v}_h\|_{H(\mathrm{div};\Omega)} \right).$$

This section is devoted to the estimation of this error. The goal is thus to show that

$$\inf_{q_h \in L^2(\Omega)} \|p - q_h\|_{L^2(\Omega)} \le ChN(p) \quad \text{and} \quad \inf_{\mathbf{v}_h \in H(\mathrm{div},\Omega)} \|\mathbf{u} - \mathbf{v}_h\|_{H(\mathrm{div};\Omega)} \le ChN(\mathbf{u}),$$

where C is a constant independent of the mesh parameter h and the particular function being approximated and $N(p)$, respectively $N(\mathbf{u})$, represents some norm of the function, p, respectively $\mathbf{u}$, being approximated. For the more classical approximation spaces, the usual technique is to define a projection operator into the approximation space and use a linear mapping to a reference element which commutes with the projection operator and then to use a scaling argument. In the present case we define projection operators, but there is no linear mapping to a reference element. Such a mapping would be trilinear and would not produce the desired scaling argument. For both the scalar and the vector functions we define a projection operator which is factored through a projection into a classical approximation space. We then obtain the standard estimate.

To calculate interpolation errors we will, following [11], define a norm on the 4 dimensional spaces $\mathbf{RTN}_0(T)$, for T a tetrahedron, by

$$|[\mathbf{v}]|_T = \int_{\partial T} |\mathbf{v}_j \cdot \mathbf{n}_T|, \quad \forall \mathbf{v} \in \mathbf{RTN}_0(T).$$

That this semi norm is in fact a norm is evident because in $\mathbf{RTN}_0(T)$, an element with zero normal component on each face of T is the zero vector function. As $\mathbf{RTN}_0(T)$ is finite dimensional, any two norms are equivalent so there exists positive constants $\alpha_0(T)$ and $\alpha_1(T)$ such that

$$\alpha_0(T) \le \frac{\|\mathbf{v}\|_{0,T}^2}{|[\mathbf{v}]|_T^2} \le \alpha_1(T), \quad \forall \mathbf{v} \in \mathbf{RTN}_0(T).$$

Definition 1 *For T a tetrahedron we define ρ_T and h_T by the radius of the inscribed sphere for the tetrahedron T and the diameter of T respectively. Following Arnold and al. [7] we introduce the notion of shape regularity. To define the composite element for a hexahedron E, we divided the hexahedron into 5 tetrahedra. There are two possible ways to decompose E into tetrahedra in such a fashion, each resulting in 5 tetrahedra. We let ρ_E be the smallest radius of the inscribed spheres for these 10 tetrahedra, and let h_E be the diameter of E. Then the shape constant of E is defined to be $\sigma_E = \dfrac{h_E}{\rho_E}$. The shape constant for a mesh $\mathcal{T}_h$ consisting of convex hexahedra is the supremum of the shape constants σ_E for $E \in \mathcal{T}_h$. A family of meshes $\{\mathcal{T}_h \,:\, h \in \mathcal{H}\}$ is said to be shape regular if the shape constants for the meshes can be uniformly bounded.*

Lemma 1 *If the family of discretizations $\{\mathcal{T}_h, h \in \mathcal{H}\}$ is shape regular, then there are constants β_0 and β_1, independent of T and h, such that $\forall T \in \widetilde{\mathcal{T}}_E,\ E \in \mathcal{T}_h,\ h \in \mathcal{H}$,*

$$\beta_0 \le \frac{\|\mathbf{v}\|_{0,T}^2 h_E^4}{|[\mathbf{v}]|_T^2 |T|} \le \beta_1. \tag{4.1}$$

Proof: The result follows from a scaling argument: if $\hat{T}$ is a reference tetrahedral element (for which we note $\hat{h} = h_{\hat{T}}, \hat{\rho} = \rho_{\hat{T}}$) and T is the image of $\hat{T}$ under a bijective affine mapping

G and if $\forall \hat{\mathbf{v}} \in \mathbf{RTN}_0(\hat{T}), \mathbf{v}$ denotes the image of $\hat{\mathbf{v}}$ under the Piola transformation, then $|[\mathbf{v}]|_T = |[\hat{\mathbf{v}}]|_{\hat{T}}$. It is also well known [6] that

$$\frac{1}{|J|\|DG^{-1}\|^2}\|\hat{\mathbf{v}}\|^2_{0,\hat{T}} \leq \|\mathbf{v}\|^2_{0,T} \leq \frac{1}{|J|}\|DG\|^2\|\hat{\mathbf{v}}\|^2_{0,\hat{T}},$$

where DG is the linear part of G and J is its determinant. Thus

$$\frac{1}{|J|\|DG^{-1}\|^2}\frac{\|\hat{\mathbf{v}}\|^2_{0,\hat{T}}}{|[\hat{\mathbf{v}}]|_{\hat{T}}} \leq \frac{\|\mathbf{v}\|^2_{0,T}}{|[\mathbf{v}]|_T} \leq \frac{1}{|J|}\|DG\|^2\frac{\|\hat{\mathbf{v}}\|^2_{0,\hat{T}}}{|[\hat{\mathbf{v}}]|_{\hat{T}}},$$

and

$$\frac{|\hat{T}|^2}{|T|^2}\frac{\rho_T^2}{\hat{h}^2}\frac{\|\hat{\mathbf{v}}\|^2_{0,\hat{T}}}{|\hat{T}||[\hat{\mathbf{v}}]|_{\hat{T}}} \leq \frac{\|\mathbf{v}\|^2_{0,T}}{|T||[\mathbf{v}]|_T} \leq \frac{|\hat{T}|^2}{|T|^2}\frac{h_T^2}{\hat{\rho}^2}\frac{\|\hat{\mathbf{v}}\|^2_{0,\hat{T}}}{|\hat{T}||[\hat{\mathbf{v}}]|_{\hat{T}}},$$

regrouping and multiplying by h_E^4

$$\frac{\rho_T^2 h_E^4}{|T|^2}\frac{|\hat{T}|\|\hat{\mathbf{v}}\|^2_{0,\hat{T}}}{\hat{h}^2|[\hat{\mathbf{v}}]|_{\hat{T}}} \leq \frac{h_E^4\|\mathbf{v}\|^2_{0,T}}{|T||[\mathbf{v}]|_T} \leq \frac{h_T^2 h_E^4}{|T|^2}\frac{|\hat{T}|\|\hat{\mathbf{v}}\|^2_{0,\hat{T}}}{\hat{\rho}^2|[\hat{\mathbf{v}}]|_{\hat{T}}}$$

using norm equality

$$\frac{\rho_T^2 h_E^4}{|T|^2}\alpha_0(\hat{T})\frac{|\hat{T}|}{\hat{h}^2} \leq \frac{h_E^4\|\mathbf{v}\|^2_{0,T}}{|T||[\mathbf{v}]|_T} \leq \frac{h_T^2 h_E^4}{|T|^2}\alpha_1(\hat{T})\frac{|\hat{T}|}{\hat{\rho}^2}$$

or

$$\frac{\rho_T^2 h_E^4}{|T|^2}\hat{\alpha}_0 \leq \frac{h_E^4\|\mathbf{v}\|^2_{0,T}}{|T||[\mathbf{v}]|_T} \leq \frac{h_T^2 h_E^4}{|T|^2}\hat{\alpha}_1,$$

where $\hat{\alpha}_0 = \alpha_0(\hat{T})\dfrac{|\hat{T}|}{\hat{h}^2}$ and $\hat{\alpha}_1 = \alpha_1(\hat{T})\dfrac{|\hat{T}|}{\hat{\rho}^2}$. To conclude it suffices to use the shape regularity of the family $\{\mathcal{T}_h : \ h \in \mathcal{H}\}$.

4.1 Local interpolation operators and error estimates. For the scalar function spaces

We denote by π_E, respectively $\tilde{\pi}_E$, the L^2-projection operator from $L^2(E)$ onto $\mathcal{M}_E$, respectively $\widetilde{\mathcal{M}}_E$:

$$\pi_E(q) = \frac{1}{|E|}\int_E q(x)dx \quad \forall E \in \mathcal{T}_E,\ \forall q \in L^2(E).$$

$$(\tilde{\pi}_E(q))_{|T} = \frac{1}{|T|}\int_T q(x)dx \quad \forall T \in \widetilde{\mathcal{T}}_E,\ \forall q \in L^2(E).$$

It is well known [2, 8] that if q is sufficiently regular then the following approximation results hold

$$\|q - \tilde{\pi}_E(q)\|_{0,E} \leq Ch|q|_{1,E}, \quad \|q - \pi_E(q)\|_{0,E} \leq Ch|q|_{1,E} \quad \forall q \in H^1(E). \tag{4.2}$$

For the vector function spaces. The Raviart-Thomas-Nedelec projection operator $\widetilde{\Pi}_E$ from $(H^1(E))^3$ onto $\widetilde{\mathcal{W}}_E$ is defined by

$$\int_{\widetilde{F}}\widetilde{\Pi}_E(\mathbf{v})\cdot\mathbf{n}_{\widetilde{F}}\,ds = \int_{\widetilde{F}}\mathbf{v}\cdot\mathbf{n}_{\widetilde{F}}\,ds \quad \forall\widetilde{F} \in \widetilde{\mathcal{F}}_E,\ \forall\mathbf{v} \in (H^1(E))^3$$

where for each $\widetilde{F} \in \widetilde{\mathcal{F}}_E$, $\mathbf{n}_{\widetilde{F}}$ is a unit vector normal to $\widetilde{F}$. It is known [2] that

$$\|\mathbf{v} - \widetilde{\Pi}_E(\mathbf{v})\|_{0,E} \leq Ch|\mathbf{v}|_{1,E} \quad \forall \mathbf{v} \in H^1(E)^3. \tag{4.3}$$

and that the interpolation operators $\widetilde{\Pi}_E$ and $\tilde{\pi}_E$ satisfy the property

$$\tilde{\pi}_E \text{div } \mathbf{v} = \text{div } \widetilde{\Pi}_E \mathbf{v} \quad \forall \mathbf{v} \in (H^1(E))^3 \tag{4.4}$$

so that

$$\|\text{div } (\mathbf{v} - \widetilde{\Pi}_E(\mathbf{v}))\|_{0,E} = \|\text{div } \mathbf{v} - \tilde{\pi}_E(\text{div } \mathbf{v})\| \leq Ch|\text{div } \mathbf{v}|_{1,E} \quad \forall \mathbf{v} \in H^1(E)^3. \tag{4.5}$$

Similarly one may define the interpolation operator $\widehat{\Pi}_E$ from $(H^1(E))^3$ onto $\widehat{\mathcal{W}}_E$ by

$$\int_{\widetilde{F}} \widehat{\Pi}_E(\mathbf{v}) \cdot \mathbf{n}_{\widetilde{F}} \, ds = \int_{\widetilde{F}} \mathbf{v} \cdot \mathbf{n}_{\widetilde{F}} \, ds \quad \forall \widetilde{F} \in \widetilde{\mathcal{F}}_E, \widetilde{F} \subset \partial E,$$

and the interpolation operator Π_E from $(H^1(E))^3$ onto $\mathcal{W}_E$ by

$$\int_F \Pi_E(\mathbf{v}) \cdot \mathbf{n}_F \, ds = \int_F \mathbf{v} \cdot \mathbf{n}_F \, ds \quad \forall F \in \mathcal{F}_E. \tag{4.6}$$

One can show that

$$\pi_E \text{div } \mathbf{v} = \text{div } \widehat{\Pi}_E \mathbf{v} = \text{div } \Pi_E \mathbf{v} \quad \forall \mathbf{v} \in (H^1(E))^3 \tag{4.7}$$

so that, if $\mathbf{v}$ is sufficiently regular,

$$\begin{aligned} \|\text{div } (\mathbf{v} - \Pi_E(\mathbf{v}))\|_{0,E}^2 &= \|\text{div } (\mathbf{v} - \widehat{\Pi}_E(\mathbf{v}))\|_{0,E}^2 = \|\text{div } \mathbf{v} - \pi_E(\text{div } \mathbf{v})\|^2 \\ &\leq Ch^2 |\text{div } \mathbf{v}|_{1,E}^2, \quad \forall \mathbf{v} \in H^1(E)^3. \end{aligned} \tag{4.8}$$

We note that $\Pi_E(\mathbf{v})$ is also determined by:

$$\forall F \in \mathcal{F}_E \quad \int_F \Pi(\mathbf{v})_E \cdot \mathbf{n}_F ds = \sum_{j=1}^{2} \int_{\widetilde{F}_j} \widehat{\Pi}_E(\mathbf{v}) \cdot \mathbf{n}_F \quad \text{where} \quad F = \widetilde{F}_1 \cup \widetilde{F}_2 \quad \text{with} \quad \widetilde{F}_1, \widetilde{F}_2 \in \widetilde{\mathcal{F}}_E.$$

To obtain an estimate for $\|\mathbf{v} - \Pi_E(\mathbf{v})\|_{0,E}$ we write

$$\|\mathbf{v} - \Pi_E(\mathbf{v})\|_{0,E}^2 \leq \|\mathbf{v} - \widetilde{\Pi}_E(\mathbf{v})\|_{0,E}^2 + \|\widetilde{\Pi}_E(\mathbf{v}) - \widehat{\Pi}_E(\mathbf{v})\|_{0,E}^2 + \|\widehat{\Pi}_E(\mathbf{v}) - \Pi_E(\mathbf{v})\|_{0,E}^2. \tag{4.9}$$

Since we have (4.3), there remains to estimate the last two terms in the right hand side.

First however, we number the faces of $F \in \mathcal{F}_E$ arbitrarily and then number the faces $\widetilde{F} \in \widetilde{\mathcal{F}}_E$ as follows: let $\widetilde{F}_i = T_i^E \cap T_5^E$ be the interior faces of T_i^E, $i = 1, \cdots, 4$, and let the remaining faces, the exterior faces, be numbered such that $F_i = \widetilde{F}_{2i+3} \cup \widetilde{F}_{2i+4}, i = 1, \cdots, 6$.

Recall that, for $i = 1, \cdots 16$, $\tilde{\omega}_i$ denotes the basis element of $\widetilde{\mathcal{W}}_E$ whose constant normal component on the face $\widetilde{F}_j$ is $\delta_{i,j}, j = 1, \cdots, 16$. Also for $i = 5, \cdots 16$, $\hat{\omega}_i$ denotes the basis element of $\widehat{\mathcal{W}}_E$ whose constant normal component on the face $\widetilde{F}_j$ is $\delta_{i,j}, j = 5, \cdots, 16$. And for $i = 1, \cdots 6$, ω_i denotes the basis element of $\mathcal{W}_E$ whose constant normal component on the face F_j is $\delta_{i,j}, j = 1, \cdots, 6$. Similarly we let χ_E be the constant function of value one on E, and for $j = 1, \cdots, 5$, let χ_j be the characteristic function of T_j^E defined on E. Then we have

$$
\begin{array}{rlll}
\widetilde{\Pi}_E \mathbf{v} = \displaystyle\sum_{i=1}^{16} \tilde{\phi}_i \tilde{\omega}_i & \text{where} & \tilde{\phi}_i = \dfrac{1}{|\widetilde{F}_i|} \displaystyle\int_{\widetilde{F}_i} \widetilde{\Pi}_E \mathbf{v} \cdot \mathbf{n}_{\widetilde{F}_i} & = \dfrac{1}{|\widetilde{F}_i|} \displaystyle\int_{\widetilde{F}_i} \mathbf{v} \cdot \mathbf{n}_{\widetilde{F}_i}, \quad i = 1, \cdots, 16 \\
\text{div } \widetilde{\Pi}_E \mathbf{v} = \displaystyle\sum_{j=1}^{5} \tilde{\psi}_j \tilde{\chi}_j & \text{where} & \tilde{\psi}_j = \dfrac{1}{|T_j^E|} \displaystyle\int_{T_j^E} \text{div } \widetilde{\Pi}_E \mathbf{v} & = \dfrac{1}{|T_j^E|} \displaystyle\int_{T_j^E} \text{div } \mathbf{v}, \quad j = 1, \cdots, 5 \\
\widehat{\Pi}_E \mathbf{v} = \displaystyle\sum_{i=5}^{16} \hat{\phi}_i \hat{\omega}_i & \text{where} & \hat{\phi}_i = \dfrac{1}{|\widetilde{F}_i|} \displaystyle\int_{\widetilde{F}_i} \widehat{\Pi}_E \mathbf{v} \cdot \mathbf{n}_{\widetilde{F}_i} & = \dfrac{1}{|\widetilde{F}_i|} \displaystyle\int_{\widetilde{F}_i} \mathbf{v} \cdot \mathbf{n}_{\widetilde{F}_i}, \quad i = 5, \cdots, 16 \\
\text{div } \widehat{\Pi}_E \mathbf{v} = \hat{\psi} \chi_E & \text{where} & \hat{\psi} = \dfrac{1}{|E|} \displaystyle\int_E \text{div } \widehat{\Pi}_E \mathbf{v} & = \dfrac{1}{|E|} \displaystyle\int_E \text{div } \mathbf{v} \\
\Pi_E \mathbf{v} = \displaystyle\sum_{i=1}^{6} \phi_i \omega_i & \text{where} & \phi_i = \dfrac{1}{|F_i|} \displaystyle\int_{F_i} \Pi_E \mathbf{v} \cdot \mathbf{n}_{F_i} & = \dfrac{1}{|F_i|} \displaystyle\int_{F_i} \mathbf{v} \cdot \mathbf{n}_{F_i}, \quad i = 1, \cdots, 6 \\
\text{div } \Pi_E \mathbf{v} = \psi \chi_E & \text{where} & \psi = \dfrac{1}{|E|} \displaystyle\int_E \text{div } \Pi_E \mathbf{v} & = \dfrac{1}{|E|} \displaystyle\int_E \text{div } \mathbf{v}.
\end{array}
\tag{4.10}
$$

Also let

$$\hat{\phi}_i = \frac{1}{|\widetilde{F}_i|} \int_{\widetilde{F}_i} \widehat{\Pi}_E \mathbf{v} \cdot \mathbf{n}_{\widetilde{F}_i} \quad (\neq \frac{1}{|\widetilde{F}_i|} \int_{\widetilde{F}_i} \mathbf{v} \cdot \mathbf{n}_{\widetilde{F}_i}), \quad i = 1, \cdots, 4.$$

Note that $\tilde{\phi}_i = \hat{\phi}_i, \ i = 5, \cdots, 16, \quad \hat{\psi} = \psi$, but $\tilde{\phi}_i \neq \hat{\phi}_i, \ i = 1, \cdots, 4$.

Since the normal components of the functions $\widehat{\Pi}_E \mathbf{v}$ and $\widetilde{\Pi}_E \mathbf{v}$ are the same on ∂E but may differ on the internal faces, we have

$$
\begin{array}{rclcrcl}
\displaystyle\int_{\partial T_j^E \setminus \widetilde{F}_j} \widetilde{\Pi}_E \mathbf{v} \cdot \mathbf{n} + \tilde{\phi}_j |\widetilde{F}_j| & = & |T_j^E| \tilde{\psi}_j, \quad j = 1, \cdots, 4, & & -\displaystyle\sum_{j=1}^{4} \tilde{\phi}_j |\widetilde{F}_j| & = & |T_5^E| \tilde{\psi}_5, \\
\displaystyle\int_{\partial T_j^E \setminus \widetilde{F}_j} \widehat{\Pi}_E \mathbf{v} \cdot \mathbf{n} + \hat{\phi}_j |\widetilde{F}_j| & = & |T_j^E| \hat{\psi}, \quad j = 1, \cdots, 4, & & -\displaystyle\sum_{j=1}^{4} \hat{\phi}_j |\widetilde{F}_j| & = & |T_5^E| \hat{\psi},
\end{array}
$$

and subtracting, we obtain

$$(\tilde{\phi}_j - \hat{\phi}_j)|\widetilde{F}_j| = |T_j^E|(\tilde{\psi}_j - \hat{\psi}), \ j = 1, \cdots, 4, \quad \sum_{j=1}^{4} (\tilde{\phi}_j - \hat{\phi}_j)|\widetilde{F}_j| = |T_5^E|(\hat{\psi} - \tilde{\psi}_5). \tag{4.11}$$

Now, to estimate the second term in the right hand side of (4.9) $\| \widetilde{\Pi}_E\mathbf{v} - \widehat{\Pi}_E\mathbf{v} \|^2_{0,E}$, we use the norm equivalence (4.1) together with (4.10), and (4.11) to obtain

$$\begin{aligned}
\| \widetilde{\Pi}_E\mathbf{v} - \widehat{\Pi}_E\mathbf{v} \|^2_{0,E} &= \sum_{j=1}^{5} \| \widetilde{\Pi}_E\mathbf{v} - \widehat{\Pi}_E\mathbf{v} \|^2_{0,T_j^E} \\
&\leq \frac{\beta_1}{h_E^4} \sum_{j=1}^{5} |T_j^E| |[\widetilde{\Pi}_E\mathbf{v} - \widehat{\Pi}_E\mathbf{v}]|^2_{T_j^E} \\
&= \frac{\beta_1}{h_E^4} \{ \sum_{j=1}^{4} |T_j^E| (\int_{\partial T_j^E} |\widetilde{\Pi}_E\mathbf{v}\cdot\mathbf{n}_{T_j^E} - \widehat{\Pi}_E\mathbf{v}\cdot\mathbf{n}_{T_j^E}|)^2 \\
&\qquad + |T_5^E| (\int_{\partial T_5^E} |\widetilde{\Pi}_E\mathbf{v}\cdot\mathbf{n}_{T_5^E} - \widehat{\Pi}_E\mathbf{v}\cdot\mathbf{n}_{T_5^E}|)^2 \} \\
&= \frac{\beta_1}{h_E^4} \{ \sum_{j=1}^{4} |T_j^E| |\tilde{\phi}_j - \hat{\phi}_j|^2 |\widetilde{F}_j|^2 + |T_5^E| (\sum_{j=1}^{4} |\tilde{\phi}_j - \hat{\phi}_j| |\widetilde{F}_j|)^2 \} \\
&\leq \frac{\beta_1}{h_E^4} \{ \sum_{j=1}^{4} |T_j^E| |\tilde{\psi}_j - \hat{\psi}|^2 |T_j^E|^2 + |T_5^E| (\sum_{j=1}^{4} |\tilde{\psi}_j - \hat{\psi}| \; |T_j^E|)^2 \} \\
&\leq \frac{\beta_1}{h_E^4} \{ \sum_{j=1}^{4} |T_j^E| |\tilde{\psi}_j - \hat{\psi}|^2 |T_j^E|^2 + 4|T_5^E| \sum_{j=1}^{4} |\tilde{\psi}_j - \hat{\psi}|^2 |T_j^E|^2 \} \\
&\leq 5\beta_1 \frac{|E|^2}{h_E^4} \sum_{j=1}^{4} |\tilde{\psi}_j - \hat{\psi}|^2 |T_j^E| \\
&= 5\beta_1 \frac{|E|^2}{h_E^4} \sum_{j=1}^{4} | \frac{1}{|T_j^E|} \int_{T_j^E} \text{div } (\widetilde{\Pi}_E\mathbf{v} - \widehat{\Pi}_E\mathbf{v}) |^2 |T_j^E| \\
&\leq 5\beta_1 \frac{|E|^2}{h_E^4} \sum_{j=1}^{4} \| \text{div } (\widetilde{\Pi}_E\mathbf{v} - \widehat{\Pi}_E\mathbf{v}) \|^2_{0,T_j^E} \\
&\leq 5\beta_1 h_E^2 \| \text{div } \widetilde{\Pi}_E\mathbf{v} - \text{div } \widehat{\Pi}_E\mathbf{v} \|^2_{0,E} \\
&\leq 5\beta_1 h_E^2 \| \text{div } \widetilde{\Pi}_h\mathbf{v} - \text{div } \widehat{\Pi}_h\mathbf{v} \|^2_{0,E} \\
&\leq 5\beta_1 h_E^2 \| \tilde{\pi}_h \text{div } \mathbf{v} - \pi_h \text{div } \mathbf{v} \|^2_{0,E} \\
&\leq 5\beta_1 h_E^2 \| \tilde{\pi}_h \text{div } \mathbf{v} - \pi_h (\tilde{\pi}_h \text{div } \mathbf{v}) \|^2_{0,E}
\end{aligned}$$

Since

$$\| (I - \pi_h)\tilde{\pi}_h \text{div } \mathbf{v} \|^2_0 \leq 2 \| \tilde{\pi}_h \text{div } \mathbf{v} \|^2_0 + 2 \| \pi_h \tilde{\pi}_h \text{div } \mathbf{v} \|^2_0 \leq 4 \| \tilde{\pi}_h \text{div } \mathbf{v} \|^2_0$$

we obtain

$$\| \widetilde{\Pi}_h\mathbf{v} - \widehat{\Pi}_h\mathbf{v} \|^2_{0,E} \leq 20\beta_1 h_E^2 \| \tilde{\pi}_h \text{div } \mathbf{v} \|^2_{0,E} \leq 20\beta_1 h_E^2 \| \text{div } \mathbf{v} \|^2_{0,E} \, . \tag{4.12}$$

To estimate the third and final term $\| \widehat{\Pi}_E\mathbf{v} - \Pi_E\mathbf{v} \|_{0,E}$ of (4.9) we use the fact that for each exterior tetrahedra $T_j^E, \quad j = 1, \cdots, 4$

$$\text{div } (\widehat{\Pi}_E\mathbf{v} - \Pi_E\mathbf{v})_{|T_j^E} = 0, \quad \int_{\partial T_j^E} (\widehat{\Pi}_E\mathbf{v} - \Pi_E\mathbf{v}) \cdot \mathbf{n}_{T_j^E} = 0, \quad j = 1, \cdots, 4.$$

Then for $\mathbf{w} = \widehat{\Pi}_E\mathbf{v} - \Pi_E\mathbf{v}$

$$\int_{\widetilde{F}_j} \mathbf{w}\cdot\mathbf{n}_{T_j^E} = -\int_{\partial T_j^E\setminus\widetilde{F}_j} \mathbf{w}\cdot\mathbf{n}_{T_j^E}, \quad j=1,\cdots,4$$

and

$$\int_{\widetilde{F}_j} |\mathbf{w}\cdot\mathbf{n}_{T_j^E}| = |\int_{\widetilde{F}_j} \mathbf{w}\cdot\mathbf{n}_{T_j^E}| = |\int_{\partial T_j^E\setminus\widetilde{F}_j} \mathbf{w}\cdot\mathbf{n}_{T_j^E}| \le \int_{\partial T_j^E\setminus\widetilde{F}_j} |\mathbf{w}\cdot\mathbf{n}_{T_j^E}|, \quad j=1,\cdots,4$$

so we obtain

$$\sum_{j=1}^{4}\int_{\widetilde{F}_j} |\mathbf{w}\cdot\mathbf{n}_{T_j^E}| \le \sum_{j=1}^{4}\int_{\partial T_j^E\setminus\widetilde{F}_j} |\mathbf{w}\cdot\mathbf{n}_{T_j^E}| = \sum_{i=5}^{16}\int_{\widetilde{F}_i} |\mathbf{w}\cdot\mathbf{n}_{\widetilde{F}_i}|.$$

Then we have

$$\begin{aligned}
\| \widehat{\Pi}_E\mathbf{v} - \Pi_E\mathbf{v} \|_{0,E}^2 &= \sum_{j=1}^{5} \| \widehat{\Pi}_E\mathbf{v} - \Pi_E\mathbf{v} \|_{0,T_j^E}^2 \\
&\le \frac{\beta_1}{h_E^4}\sum_{j=1}^{5} |T_j^E|\,|[\widehat{\Pi}_E\mathbf{v} - \Pi_E\mathbf{v}]|_{\partial T_j^E}^2 \\
&\le \frac{\beta_1}{h_E^4}|E|\{\sum_{j=1}^{4}(\int_{\partial T_j^E} |(\widehat{\Pi}_E\mathbf{v} - \Pi_E\mathbf{v})\cdot\mathbf{n}_{T_j^E}|)^2 \\
&\qquad +(\int_{\partial T_5^E} |(\widehat{\Pi}_E\mathbf{v} - \Pi_E\mathbf{v})\cdot\mathbf{n}_{T_5^E}|)^2\} \\
&\le \frac{\beta_1}{h_E^4}|E|\{8\sum_{i=1}^{4}(\int_{\widetilde{F}_i} |(\widehat{\Pi}_E\mathbf{v} - \Pi_E\mathbf{v})\cdot\mathbf{n}_{\widetilde{F}_i}|)^2 \\
&\qquad +4\sum_{i=5}^{16}(\int_{\widetilde{F}_i} |(\widehat{\Pi}_E\mathbf{v} - \Pi_E\mathbf{v})\cdot\mathbf{n}_{\widetilde{F}_i}|)^2\} \\
&\le C\frac{\beta_1}{h_E^4}|E|\{\sum_{i=5}^{16}(\int_{\widetilde{F}_i} |(\widehat{\Pi}_E\mathbf{v} - \Pi_E\mathbf{v})\cdot\mathbf{n}_{\widetilde{F}_i}|)^2\} \\
&\le C\frac{\beta_1}{h_E^4}|E|\{\sum_{i=5}^{16} |\widetilde{F}_i|\, \| (\widehat{\Pi}_E\mathbf{v} - \Pi_E\mathbf{v})\cdot\mathbf{n}_{\widetilde{F}_i} \|_{0,\widetilde{F}_i}^2\} \\
&\le Ch_E\{\sum_{i=5}^{16} \| (\widehat{\Pi}_E\mathbf{v} - \mathbf{v})\cdot\mathbf{n}_{\widetilde{F}_i} \|_{0,\widetilde{F}_i}^2 + \sum_{l=1}^{6} \| (\Pi_E\mathbf{v} - \mathbf{v})\cdot\mathbf{n}_{F_l} \|_{0,F_l}^2\}
\end{aligned} \tag{4.13}$$

Since $\widehat{\Pi}_E$ and Π_E can be interpreted as L^2 projections of the scalar function $\mathbf{v}\cdot\mathbf{n}$ on the faces $\widetilde{F}_i$ and F_i respectively, we have the approximation estimates

$$\begin{aligned}
\| (\widehat{\Pi}_E\mathbf{v} - \mathbf{v})\cdot\mathbf{n}_{\widetilde{F}_i} \|_{0,\widetilde{F}_i} &\le Ch^{1/2} \| \mathbf{v} \|_{1/2,\widetilde{F}_i} \quad \forall \mathbf{v}\in (H^1(E))^3, \\
\| (\Pi_E\mathbf{v} - \mathbf{v})\cdot\mathbf{n}_{F_l} \|_{0,F_l} &\le Ch^{1/2} \| \mathbf{v} \|_{1/2,F_l} \quad \forall \mathbf{v}\in (H^1(E))^3.
\end{aligned}$$

From a standard trace theorem [9] $\| \mathbf{v} \|_{1/2,\widetilde{F}_i} \leq C \| \mathbf{v} \|_{1,E}$, so

$$\begin{aligned}\sum_{i=5}^{16} \| (\widehat{\Pi}_E \mathbf{v} - \mathbf{v}) \cdot \mathbf{n}_{\widetilde{F}_i} \|_{0,\widetilde{F}_i}^2 + \sum_{l=1}^{6} \| (\Pi_E \mathbf{v} - \mathbf{v}) \cdot \mathbf{n}_{F_l} \|_{0,F_l}^2 &\leq Ch \sum_{i=5}^{16} \| \mathbf{v} \|_{1/2,\widetilde{F}_i}^2 + Ch \sum_{l=1}^{6} \| \mathbf{v} \|_{1/2,F_l}^2 \\ &\leq Ch \| \mathbf{v} \|_{1,E}^2\end{aligned}$$

and we obtain

$$\| \widehat{\Pi}_E \mathbf{v} - \Pi_E \mathbf{v} \|_{0,E}^2 \leq Ch^2 \| \mathbf{v} \|_{1,E}^2 . \tag{4.14}$$

Now summing up the three terms (4.3), (4.12) and (4.14) we obtain

$$\begin{aligned}\| \mathbf{v} - \Pi_E \mathbf{v} \|_{0,E}^2 &= \| \mathbf{v} - \widetilde{\Pi}_E \mathbf{v} \|_{0,E}^2 + \| \widetilde{\Pi}_E \mathbf{v} - \widehat{\Pi}_E \mathbf{v} \|_{0,E}^2 + \| \widehat{\Pi}_E \mathbf{v} - \Pi_E \mathbf{v} \|_{0,E}^2 \\ &\leq Ch^2 \left(|\mathbf{v}|_{1,E}^2 + \| \operatorname{div} \mathbf{v} \|_{0,E}^2 + \| \mathbf{v} \|_{1,E}^2 \right) \\ &\leq Ch^2 \| \mathbf{v} \|_{1,E}^2 \end{aligned} \tag{4.15}$$

and by using (4.8) we obtain that

$$\|\mathbf{v} - \Pi_E(\mathbf{v})\|_{H(\operatorname{div};E)}^2 \leq Ch^2 (\| \operatorname{div} \mathbf{v} \|_{1,E}^2 + \| \mathbf{v} \|_{1,E}^2). \tag{4.16}$$

4.2 Global interpolation error estimates

We define π_h to be the L^2- projection operator from $L^2(\Omega)$ onto $\mathcal{M}_h$

$$\pi_h(q)(x) = \pi_E(q)(x) \quad if \quad x \in E,$$

and the interpolation operator Π_h from $(H^1(\Omega))^3$ onto $\mathcal{W}_h$ by

$$\Pi_h(\mathbf{v})(x) = \Pi_E(\mathbf{v})(x) \quad if \quad x \in E.$$

By summing (4.2) over the cells $E \in \mathcal{T}_h$ we obtain

$$\|q - \pi_h(q)\|_{0,\Omega}^2 \leq Ch^2 |q|_{1,\Omega}^2.$$

Then summing (4.16) over all the cells E gives

$$\|\mathbf{v} - \Pi_h(\mathbf{v})\|_{H(\operatorname{div};\Omega)}^2 \leq Ch^2 (\| \operatorname{div} \mathbf{v} \|_{1,\Omega}^2 + \| \mathbf{v} \|_{1,\Omega}^2).$$

Finally, we obtain that approximation errors are of order one:

$$\|q - \pi_h(q)\|_{0,\Omega}^2 + \|\mathbf{v} - \Pi_h(\mathbf{v})\|_{H(\operatorname{div};\Omega)}^2 \leq Ch^2 \left(|q|_{1,\Omega}^2 + \| \operatorname{div} \mathbf{v} \|_{1,\Omega}^2 + \| \mathbf{v} \|_{1,\Omega}^2 \right)$$

5 Numerical experiment

First we present numerical convergence results for the analytical solution

$$p = sin(\pi x) sin(\pi y) sin(\pi z) + x(1-x) y^2 (1-y)^2 z(1-z)$$

on meshes which are deformations of a $n \times n \times n$ uniform cubic mesh, for $n = 4, 8, 16, 32, 64$.

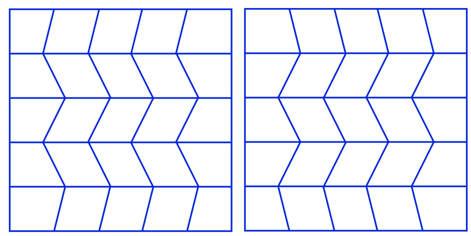


Figure 5.1: Horizontal cross sections $z = kh_z$, $z = (k+1)h_z$ in a deformed mesh ($n = 5$).

The deformation consists in moving the vertices in the horizontal plane as shown in Fig. 5.1 in order to obtain for the cells the form of truncated pyramids. Figure 5.2 shows $8\times 8\times 8$ deformed meshes with such increasing deformations.

Tables 1, 2, 3, 4 show errors for the pressure and the velocity using respectively the Raviart-Thomas-Nédélec (RTN) and the Kuznetsov-Repin (KR) mixed finite elements. Actually we are showing the errors between the calculated solution $(p_h, \mathbf{u}_h)$ and the interpolates of the analyticale solution $(\pi_h(p), \Pi_h(\mathbf{u}))$ in order to not have to use integration formulas in the calculation of the errors. These errors usually show superconvergence as we can see for the cubic meshes – order 2 instead of 1 for both pressure and velocity.

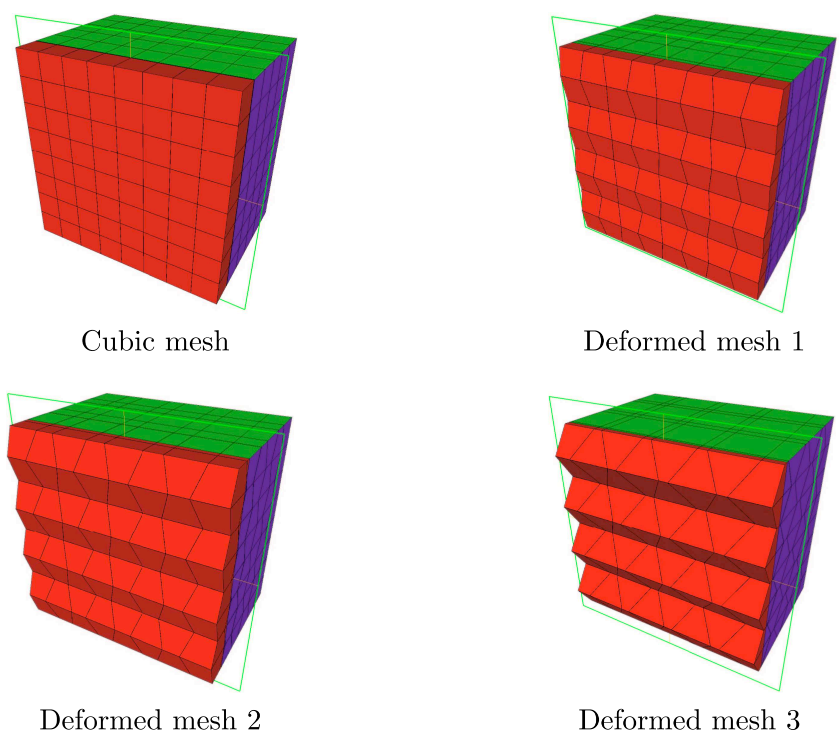


Figure 5.2: An $8\times$ 8×8 cubic mesh with 3 deformed meshes with increasing deformation from this cubic mesh.

These tables confirm theoretical results stating that the RTN method is not converging on general hexahedrons while the KR method is. Note that the order of convergence for

	RTN finite element				KR finite element			
n	$\|\| p_h - \pi_h p \|\|_{0,\Omega}$		$\|\| \mathbf{u}_h - \Pi_h \mathbf{u} \|\|_{0,\Omega}$		$\|\| p_h - \pi_h p \|\|_{0,\Omega}$		$\|\| \mathbf{u}_h - \Pi_h \mathbf{u} \|\|_{0,\Omega}$	
	error	rate	error	rate	error	rate	error	rate
4	0.0164		0.0011		0.0164		0.0012	
8	0.0044	1.88	0.0002	2.15	0.0044	1.88	0.0003	2.06
16	0.0011	1.97	6.0e-5	2.00	0.0011	1.97	7.2e-5	1.97
32	0.0003	1.99	1.5e-5	2.00	0.0003	1.99	1.8e-5	2.00
64	7.1e-5	2.00	3.8e-6	1.98	7.1e-5	2.00	4.5e-6	2.00

Table 1: Pressure and velocity errors for RTN and KR mixed finite elements on the sequence of cubic meshes.

	RTN finite element				KR finite element			
n	$\|\| p_h - \pi_h p \|\|_{0,\Omega}$		$\|\| \mathbf{u}_h - \Pi_h \mathbf{u} \|\|_{0,\Omega}$		$\|\| p_h - \pi_h p \|\|_{0,\Omega}$		$\|\| \mathbf{u}_h - \Pi_h \mathbf{u} \|\|_{0,\Omega}$	
	error	rate	error	rate	error	rate	error	rate
4	0.0172		0.0788		0.0161		0.0395	
8	0.0056	1.6	0.0405	0.96	0.0042	1.92	0.0132	1.58
16	0.0029	0.62	0.0263	0.62	0.0011	1.99	0.0042	1.64
32	0.0024	0.3	0.0234	0.16	2.6e-4	2.	0.0014	1.62
64	0.0023	0.08	0.0229	0.03	6.5e-5	2.01	4.6e-4	1.32

Table 2: Pressure and velocity errors for RTN and KR mixed finite elements on the sequence of deformed meshes 1.

	RTN finite element				KR finite element			
n	$\|\| p_h - \pi_h p \|\|_{0,\Omega}$		$\|\| \mathbf{u}_h - \Pi_h \mathbf{u} \|\|_{0,\Omega}$		$\|\| p_h - \pi_h p \|\|_{0,\Omega}$		$\|\| \mathbf{u}_h - \Pi_h \mathbf{u} \|\|_{0,\Omega}$	
	error	order	error	rate	error	rate	error	rate
4	0.0206		0.1773		0.0152		0.0785	
8	0.0119	0.79	0.1160	0.61	0.0036	2.07	0.0265	1.56
16	0.0100	0.25	0.0994	0.22	8.3e-4	2.10	0.0088	1.57
32	0.0095	0.08	0.0967	0.04	2.0e-5	2.06	0.0032	1.48
64	0.0093	0.02	0.0961	0.01	4.9e-5	2.03	0.0013	1.32

Table 3: Pressure and velocity errors for RTN and KR mixed finite elements on the sequence of deformed meshes 2

	RTN finite element				KR finite element			
n	$\|\| p_h - \pi_h p \|\|_{0,\Omega}$		$\|\| \mathbf{u}_h - \Pi_h \mathbf{u} \|\|_{0,\Omega}$		$\|\| p_h - \pi_h p \|\|_{0,\Omega}$		$\|\| \mathbf{u}_h - \Pi_h \mathbf{u} \|\|_{0,\Omega}$	
	error	rate	error	rate	error	rate	error	rate
4	0.04736		0.5234		0.0135		0.1671	
8	0.04798	-0.01	0.4770	0.13	0.0023	2.55	0.0699	1.25
16	0.04588	0.06	0.4727	0.01	3.7e-4	2.68	0.0319	1.12
32	0.04487	0.03	0.4733	-0.00	7.0e-5	2.36	0.0155	1.04
64	0.04457	0.01	0.4731	5.e-4	1.6e-5	2.12	7.7e-3	1.01

Table 4: Pressure and velocity errors for RTN and KR mixed finite elements on the sequence of deformed meshes 3

the velocity is decreasing with the deformation. However, even for deformed meshes 3 which correspond to very large deformations, convergence is maintained. This shows the robustness of the method.

Once the Kuznetsov-Repin mixed finite element method was proven to be convergent, it was used to calculate the pressure and velocity field around a nuclear waste disposal [18]. Figure 5.3 shows the domain of calculation on the top. It is made up of 13 geological subdomains with permeabilities changing with up to three orders of magnitude from one subdomain to the other.

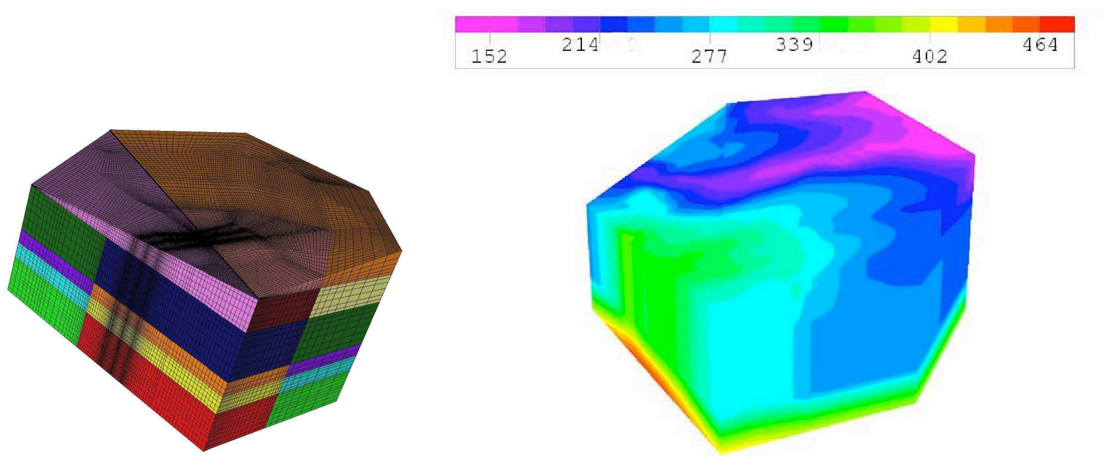


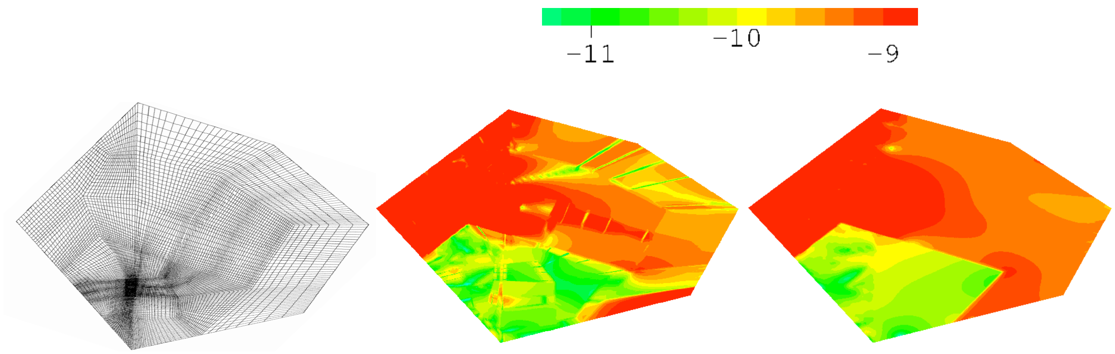
Figure 5.3: The domain of calculation (top) and the calculated pressure field (bottom)

The permeabilities of the geological layers have two principal characteristics: on the one hand, the values are extremely small, and on the other hand they are particularly heterogeneous. In Figure 5.3 top the mesh is shown. It was provided by engineers from Andra (http://www.andra.fr) and is made of about 500,000 hexahedrons which for the most part are not parallelepipeds. On the right in Figure 5.3 the calculated pressure field calculated with KR mixed finite elements is shown. Both figures are blown up 30 times in the z-direction in order to visualize the results.

However, since the Darcy velocity is actually the important quantity that is needed for the transport, we show in Fig. 5.4 the norm of the velocity on an horizontal cross section, the velocity calculated with RTN finite elements (center) and KR mixed finite elements (right) and the corresponding cross section of the mesh (left). The scale on the color bar corresponds to powers of 10. As one can observe, there are significant differences in the calculated velocity. In particular the norm of the velocity calculated with RTN mixed finite elements show a rough behavior which is clearly unphysical for regions with constant permeabilities. This will necessarily have a strong impact when this velocity will be used in transport calculations.

6 Conclusion

We constructed a new mixed finite element for general hexahedral grids based on Kuznetsov's and Repin's general procedure for composite mixed finite elements. This new mixed finite element provides an elegant and simple way to implement mixed finite elements for general



Cross section of the mesh RTN finite element KR finite element

Figure 5.4: Horizontal cross section of the norm of the Darcy velocity for the RTN (left) and KR (right) mixed finite elements.

hexahedral discretizations. Theoretical convergence was proven and numerical convergence was observed. The method is applied to the calculation of a Darcy velocity which will be used for the simulation of the transport of radionucides around a storage site.

We pointout, however, that this element is not adapted to general distorted cubes with nonplanar faces.

APPENDIX: Basis functions for the new finite element

6.1 Geometry

The faces of an hexahedron E are denoted by $FE_i, i = 1, \ldots .6$. Their numbering is shown in Fig. 6.1 and their area is denoted by $|FE_i|$. The volume of the hexahedron is denoted by $|E|$. The numbering of the vertices in the hexahedron and in the tetrahedrons, and the numbering of the tetrahedrons are shown in Tables 5 and 6.

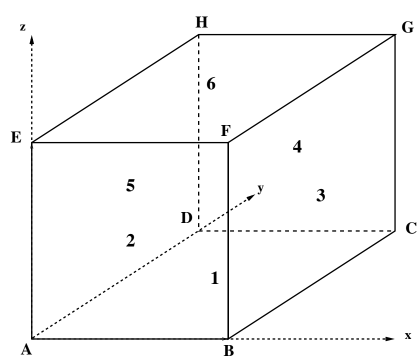


Figure 6.1: Reference hexaedron with a numbering of its faces

Vertices	A	B	C	D	E	F	G	H
Vertex number	1	2	3	4	5	6	7	8

Table 5: Vertex numbering in the reference hexahedron of Fig. 6.1

.

Tetrahedron 1	Vertices	A	B	D	E
Tetrahedron 2	Vertices	F	G	B	E
Tetrahedron 3	Vertices	C	D	B	G
Tetrahedron 4	Vertices	H	G	E	D
Tetrahedron 5	Vertices	B	D	E	G

Table 6: Definition of the 5 tetrahedrons in the division shown in Fig. 6.1

.

6.2 Basis functions

In each tetrahedron $T_\ell, \ell = 1, \cdots, 5$, the faces are denoted $FT_{\ell,i}, i = 1, \ldots, 4$ and numbered in order that it has the same number as the vertex facing it. Therefore, if we denote $\mathbf{s}^{\ell,i}, i = 1, \ldots, 4$, the vertices of the tetrahedron T_ℓ, a standard RTN_0 basis in this tetrahedron is

$$\mathbf{w}_{T_\ell,i} = \frac{|FT_{\ell,i}|}{3|T_\ell|}(\mathbf{x} - \mathbf{s}^{\ell,i}), \quad i = 1, \ldots, 4, \ell = 1, \cdots, 5$$

where $|FT_{\ell,i}|$ denotes the area of the face $FT_{\ell,i}$ and $|T_\ell|$ the volume of the tetrahedron T_ℓ. These basis functions are such that

$$\mathbf{w}_{T_\ell,i} \cdot \mathbf{n}_j = \delta_{ij}, \quad i, j = 1, 2, 3, 4$$

where $\mathbf{n}_j$ denotes the unit outward normal to the face $FT_{\ell,j}$. Therefore they satisfy also

$$\text{div } \mathbf{w}_{T_\ell,i} = \frac{|FT_{\ell,i}|}{|T_\ell|}, \quad i = 1, \ldots, 4, \ell = 1, \cdots, 5$$

A basis for the new mixed finite element, denoted for a given hexahedron E by $\mathbf{w}_{Ei}, i = 1, \ldots, 6$, can be defined as

$$\int_{FE_j} \mathbf{w}_{Ei} \cdot \mathbf{n}_j = \delta_{ij}, \quad i, j = 1, \cdots, 6.$$

They must be written tetrahedron by tetrahedron and the restriction of $\mathbf{w}_{Ei}$ to the ℓth tetrahedron is denoted by $\mathbf{w}_{Ei,\ell}, i = 1, \ldots, 6, \ell = 1, \ldots, 5$. These basis functions are

Tetrahedron 1:

$$
\begin{aligned}
\mathbf{w}_{E1,1} &= \left(\mathbf{w}_{T_{1,4}} + \frac{|T_1||FE_1| - |FT_{1,4}||E|}{|FT_{1,1}||E|}\mathbf{w}_{T_{1,1}}\right)/|FE_1| \\
\mathbf{w}_{E2,1} &= \left(\mathbf{w}_{T_{1,3}} + \frac{|T_1||FE_2| - |FT_{1,3}||E|}{|FT_{1,1}||E|}\mathbf{w}_{T_{1,1}}\right)/|FE_2| \\
\mathbf{w}_{E3,1} &= \frac{|T_1|}{|FT_{1,1}||E|}\mathbf{w}_{T_{1,1}} \\
\mathbf{w}_{E4,1} &= \frac{|T_1|}{|FT_{1,1}||E|}\mathbf{w}_{T_{1,1}} \\
\mathbf{w}_{E5,1} &= \left(\mathbf{w}_{T_{1,2}} + \frac{|T_1||FE_5| - |FT_{1,2}||E|}{|FT_{1,1}||E|}\mathbf{w}_{T_{1,1}}\right)/|FE_5| \\
\mathbf{w}_{E6,1} &= \frac{|T_1|}{|FT_{1,1}||E|}\mathbf{w}_{T_{1,1}}
\end{aligned}
$$

Tetrahedron 2:

$$
\begin{aligned}
\mathbf{w}_{E1,2} &= \frac{|T_2|}{|FT_{2,1}||E|}\mathbf{w}_{T_{2,1}} \\
\mathbf{w}_{E2,2} &= \left(\mathbf{w}_{T_{2,2}} + \frac{|T_2||FE_2| - |FT_{2,2}||E|}{|FT_{2,1}||E|}\mathbf{w}_{T_{2,1}}\right)/|FE_2| \\
\mathbf{w}_{E3,2} &= \left(\mathbf{w}_{T_{2,4}} + \frac{|T_2||FE_3| - |FT_{2,4}||E|}{|FT_{2,1}||E|}\mathbf{w}_{T_{2,1}}\right)/|FE_3| \\
\mathbf{w}_{E4,2} &= \frac{|T_2|}{|FT_{2,1}||E|}\mathbf{w}_{T_{2,1}} \\
\mathbf{w}_{E5,2} &= \frac{|T_2|}{|FT_{2,1}||E|}\mathbf{w}_{T_{2,1}} \\
\mathbf{w}_{E6,2} &= \left(\mathbf{w}_{T_{2,3}} + \frac{|T_2||FE_6| - |FT_{2,3}||E|}{|FT_{2,1}||E|}\mathbf{w}_{T_{2,1}}\right)/|FE_6|
\end{aligned}
$$

Tetrahedron 3:

$$
\begin{aligned}
\mathbf{w}_{E1,3} &= \left(\mathbf{w}_{T_{3,4}} + \frac{|T_3||FE_1| - |FT_{3,4}||E|}{|FT_{3,1}||E|}\mathbf{w}_{T_{3,1}}\right)/|FE_1| \\
\mathbf{w}_{E2,3} &= \frac{|T_3|}{|FT_{3,1}||E|}\mathbf{w}_{T_{3,1}} \\
\mathbf{w}_{E3,3} &= \left(\mathbf{w}_{T_{3,2}} + \frac{|T_3||FE_3| - |FT_{3,2}||E|}{|FT_{3,1}||E|}\mathbf{w}_{T_{3,1}}\right)/|FE_3| \\
\mathbf{w}_{E4,3} &= \left(\mathbf{w}_{T_{3,3}} + \frac{|T_3||FE_4| - |FT_{3,3}||E|}{|FT_{3,1}||E|}\mathbf{w}_{T_{3,1}}\right)/|FE_4| \\
\mathbf{w}_{E5,3} &= \frac{|T_3|}{|FT_{3,1}||E|}\mathbf{w}_{T_{3,1}} \\
\mathbf{w}_{E6,3} &= \frac{|T_3|}{|FT_{3,1}||E|}\mathbf{w}_{T_{3,1}}
\end{aligned}
$$

Tetrahedron 4:

$$
\begin{aligned}
\mathbf{w}_{E1,4} &= \frac{|T_4|}{|FT_{4,1}||E|}\mathbf{w}_{T_4,1}\\
\mathbf{w}_{E2,4} &= \frac{|T_4|}{|FT_{4,1}||E|}\mathbf{w}_{T_4,1}\\
\mathbf{w}_{E3,4} &= \frac{|T_4|}{|FT_{4,1}||E|}\mathbf{w}_{T_4,1}\\
\mathbf{w}_{E4,4} &= \left(\mathbf{w}_{T_4,3} + \frac{|T_4||FE_4| - |FT_{4,3}||E|}{|FT_{4,1}||E|}\mathbf{w}_{T_4,1}\right)/|FE_4|\\
\mathbf{w}_{E5,4} &= \left(\mathbf{w}_{T_4,2} + \frac{|T_4||FE_5| - |FT_{4,2}||E|}{|FT_{4,1}||E|}\mathbf{w}_{T_4,1}\right)/|FE_5|\\
\mathbf{w}_{E6,4} &= \left(\mathbf{w}_{T_4,4} + \frac{|T_4||FE_6| - |FT_{4,4}||E|}{|FT_{4,1}||E|}\mathbf{w}_{T_4,1}\right)/|FE_6|
\end{aligned}
$$

Tetrahedron 5:

$$
\begin{aligned}
\mathbf{w}_{E1,5} &= \left(-\frac{|T_1||FE_1| - |FT_{1,4}||E|}{|FT_{5,4}||E|}\mathbf{w}_{T_5,4} \quad -\frac{|T_2||FE_1|}{|FT_{5,2}||E|}\mathbf{w}_{T_5,2}\right.\\
&\quad \left. -\frac{|T_3||FE_1| - |FT_{3,4}||E|}{|FT_{5,3}||E|}\mathbf{w}_{T_5,3} \quad -\frac{|T_4||FE_1|}{|FT_{5,1}||E|}\mathbf{w}_{T_5,1}\right)/|FE_1|\\
\mathbf{w}_{E2,5} &= \left(-\frac{|T_1||FE_2| - |FT_{1,3}||E|}{|FT_{5,4}||E|}\mathbf{w}_{T_5,4} \quad -\frac{|T_2||FE_2| - |FT_{2,2}||E|}{|FT_{5,2}||E|}\mathbf{w}_{T_5,2}\right.\\
&\quad \left. -\frac{|T_3||FE_2|}{|FT_{5,3}||E|}\mathbf{w}_{T_5,3} \quad -\frac{|T_4||FE_2|}{|FT_{5,1}||E|}\mathbf{w}_{T_5,1}\right)/|FE_2|\\
\mathbf{w}_{E3,5} &= \left(-\frac{|T_1||FE_3|}{|FT_{5,4}||E|}\mathbf{w}_{T_5,4} \quad -\frac{|T_2||FE_3| - |FT_{2,3}||E|}{|FT_{5,2}||E|}\mathbf{w}_{T_5,2}\right.\\
&\quad \left. -\frac{|T_3||FE_3| - |FT_{3,2}||E|}{|FT_{5,3}||E|}\mathbf{w}_{T_5,3} \quad -\frac{|T_4||FE_3|}{|FT_{5,1}||E|}\mathbf{w}_{T_5,1}\right)/|FE_3|\\
\mathbf{w}_{E4,5} &= \left(-\frac{|T_1||FE_4|}{|FT_{5,4}||E|}\mathbf{w}_{T_5,4} \quad -\frac{|T_2||FE_4|}{|FT_{5,2}||E|}\mathbf{w}_{T_5,2}\right.\\
&\quad \left. -\frac{|T_3||FE_4| - |FT_{3,3}||E|}{|FT_{5,3}||E|}\mathbf{w}_{T_5,3} \quad -\frac{|T_4||FE_4| - |FT_{4,3}||E|}{|FT_{5,1}||E|}\mathbf{w}_{T_5,1}\right)/|FE_4|\\
\mathbf{w}_{E5,5} &= \left(-\frac{|T_1||FE_5| - |FT_{1,2}||E|}{|FT_{5,4}||E|}\mathbf{w}_{T_5,4} \quad -\frac{|T_2||FE_5|}{|FT_{5,2}||E|}\mathbf{w}_{T_5,2}\right.\\
&\quad \left. -\frac{|T_3||FE_5|}{|FT_{5,3}||E|}\mathbf{w}_{T_5,3} \quad -\frac{|T_4||FE_5| - |FT_{4,3}||E|}{|FT_{5,1}||E|}\mathbf{w}_{T_5,1}\right)/|FE_5|\\
\mathbf{w}_{E6,5} &= \left(-\frac{|T_1||FE_6|}{|FT_{5,4}||E|}\mathbf{w}_{T_5,4} \quad -\frac{|T_2||FE_6| - |FT_{2,3}||E|}{|FT_{5,2}||E|}\mathbf{w}_{T_5,2}\right.\\
&\quad \left. -\frac{|T_3||FE_6|}{|FT_{5,3}||E|}\mathbf{w}_{T_5,3} \quad -\frac{|T_4||FE_6| - |FT_{4,4}||E|}{|FT_{5,1}||E|}\mathbf{w}_{T_5,1}\right)/|FE_6|
\end{aligned}
$$

7 Aknowledgments

We would like to express our gratitude to Loïc Maréchal who introduced us to the world of hexahedral computing.

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